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LETTER TO THE EDITOR

The isotropic $O(3)$ model and the Wolff representation

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Abstract. For the isotropic $O(3)$ model we prove that percolation in the Wolff representation is a necessary and sufficient condition for positivity of the spontaneous magnetization.

The distinguishing feature of Wolff's cluster algorithm [W] for models with $O(n)$ symmetry is the apparent fact that the associated clusters percolate precisely at the point of the phase transition. Although there has been some theoretical understanding of this phenomenon [A, PS, LH] only recently has a definitive theorem been established [C] and only for the case $n = 2$. The purpose of this note is to extend these results at least as far as $n = 3$. A concise statement will be presented after some definitions and notation.

Consider the standard classical Heisenberg ferromagnet; let $\mathcal{G}$ denote a graph with bonds $\mathbb{B}$ and sites $\mathbb{S}$. The Hamiltonian is given by

$$\mathcal{H} = -\sum_{\langle i,j\rangle\in\mathbb{B}} J_{i,j}\boldsymbol{s}_i\cdot\boldsymbol{s}_j - \sum_{i\in\mathbb{S}}\boldsymbol{s}_i\cdot\boldsymbol{h}_i \tag{1}$$

where $\boldsymbol{s}_i$ is a unit vector in $\mathbb{R}^3$, the $\boldsymbol{h}_i$ any vectors in $\mathbb{R}^3$ and the $J_{i,j} > 0$. The partition function is given by

$$\mathcal{Z}_{\mathcal{G}}(\beta) = \int \mathrm{d}^{|\mathbb{S}|}s\, \mathrm{e}^{-\beta\mathcal{H}}$$

where the integrations are with respect to the Haar measure on the sphere.

To define the Wolff representation, we single out the z-direction as the focus of our attention. Let us write $\boldsymbol{s}_i = (a_i\hat{x}_i, a_i\hat{y}_i, b_i\sigma_i)$ where b_i is the absolute value of the projection of $\boldsymbol{s}_i$ onto the z-axis, σ_i is an Ising variable, $a_i = \sqrt{1-b_i^2}$ and the $(\hat{x}_i, \hat{y}_i)$ are the usual $O(2)$ variables. Using $\sigma_i\sigma_j = 2\delta_{\sigma_i\sigma_j} - 1$ and temporarily setting $\boldsymbol{h}_i \equiv 0$, we have

$$\mathcal{Z}_{\mathcal{G}}(\beta) = \int \mathrm{d}^{|\mathbb{S}|}b \prod_{\langle i,j\rangle} \mathrm{e}^{\beta J_{i,j}b_ib_j}\mathcal{Z}_{\underline{b}}^{[I]}(2\beta)\mathcal{Z}_{\underline{a}}^{[XY]}(\beta) \tag{2}$$

where, on the right-hand side, the dependence on $\mathcal{G}$ and the $(J_{i,j})$ has been suppressed and the various terms are defined as follows. The quantity $\underline{b}$ is a configuration of the b, $\underline{b} = (b_i | i \in \mathbb{S})$—and similarly for $\underline{a}$—the object $\mathcal{Z}_{\underline{b}}^{[I]}$ is the Ising partition function written in Potts form

$$\mathcal{Z}_{\underline{b}}^{[I]}(2\beta) = \sum_{\sigma_i}\prod_{\langle i,j\rangle\in\mathbb{B}} \mathrm{e}^{2\beta b_ib_jJ_{i,j}(\delta_{\sigma_i\sigma_j}-1)}$$

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and $\mathcal{Z}_{\underline{a}}^{[XY]}(\beta)$ is the XY partition function with couplings $J_{i,j}a_i a_j$ at inverse temperature β.

We expand $\mathcal{Z}_{\underline{b}}^{[I]}(2\beta) \equiv \sum_{\omega\subset\mathbb{B}} B_{\underline{b},2\beta}(\omega)2^{c(\omega)}$ in the standard FK representation. Here

$$B_{\underline{b},2\beta}(\omega) = \prod_{\langle i,j\rangle\in\omega} 1 - \mathrm{e}^{-2\beta b_i b_j} \prod_{\langle i,j\rangle\notin\omega} \mathrm{e}^{-2\beta b_i b_j}$$

and $c(\omega)$ is the number of connected components of the bond configuration ω. Hence a joint measure is defined on configurations $(\underline{b},\omega)$ of 'spin projections' and bonds. This is the Wolff measure and is denoted by $V_\beta^W(-)$. The bond marginal is denoted by $\mathbb{M}_\beta(-)$ and the $\underline{b}$-marginal by $\rho_\beta(-)$.

For realistic problems, e.g. on $\mathbb{Z}^d$, we must discuss boundary conditions. For $\Lambda\subset\mathbb{Z}^d$ we will need to consider free boundary conditions (for which nothing has to be said) and *wired* boundary conditions meaning $b_i\sigma_i \equiv 1$ for all $i\in\partial\Lambda$. In general, these measures with some specification * on the boundary will be denoted by a superscripted * i.e. $\mathbb{M}_\beta^*(-)$ and $\rho_\beta^*(-)$; our notation for free and wired will be f and w. Our principal result can now be stated.

Theorem 1. Let Λ_k denote a thermodynamic sequence of finite boxes; $0\in\Lambda_{k-1}\subset\Lambda_k\subset\mathbb{Z}^d$ with $\Lambda_k \nearrow \mathbb{Z}^d$. Let $\Pi_k(\beta)$ denote the probability, in $\mathbb{M}^w_{\beta;\Lambda_k}(-)$ that the origin is connected to the boundary $\partial\Lambda_k$. Then the limit

$$\Pi_\infty(\beta) = \lim_{k\to\infty} \Pi_k(\beta)$$

exists and satisfies

$$\Pi_\infty(\beta) \geqslant m(\beta) \geqslant K\,\Pi_\infty(\beta)$$

where $m(\beta)$ is the spontaneous magnetization. Here K is a finite and non-singular function of temperature and coordination number. Hence, the magnetization is positive iff there is percolation as defined by the condition $\Pi_\infty(\beta) > 0$. Furthermore, in the high-temperature phase, the spin–spin correlation function and the magnetic susceptibility enjoy similar bounds by appropriate quantities in the graphical representation. In particular, the susceptibility is bounded above and below by 'constants' times the average of the size of the cluster at the origin.

The proof of theorem 1 is a consequence of the following technical lemma.

Lemma 2. On a finite graph $\mathcal{G}$, the measure $\rho_\beta(-)$ is strong FKG.

Proof. Let $u,v\in\mathbb{S}$ and ϵ_u, ϵ_v denote numbers (that may be regarded as small). For fixed $\underline{b}$, let δ_u denote the configuration that agrees with $\underline{b}$ at each site save u where it takes on the value $b_u+\epsilon_u$. The configuration δ_v is defined similarly and it is assumed, without loss of generality, that $b_u+\epsilon_u$, etc is less than one. It is sufficient to show

$$\rho_\beta(\underline{b}\vee\delta_u\vee\delta_v)\rho_\beta(\underline{b}) \geqslant \rho_\beta(\underline{b}\vee\delta_u)\rho_\beta(\underline{b}\vee\delta_v)$$

where, with apologies, we use the same notation for the density and the measure. Indeed, a moment of contemplation reveals that it is in fact sufficient to verify the above to lowest non-vanishing order in ϵ_u, ϵ_v. It has been shown [C, equation (A.5)] that

$$\mathrm{e}^{\beta J_{u,v}\epsilon_u\epsilon_v}\mathcal{Z}^{[I]}_{\underline{b}\vee\delta_u\vee\delta_v}(2\beta)\mathcal{Z}^{[I]}_{\underline{b}}(2\beta) \geqslant \mathcal{Z}^{[I]}_{\underline{b}\vee\delta_u}(2\beta)\mathcal{Z}^{[I]}_{\underline{b}\vee\delta_v}(2\beta). \tag{3}$$

Thus, it is sufficient to establish $\mathcal{Z}^{[XY]}_{\underline{a}(\underline{b}\vee\delta_u\vee\delta_v)}(\beta)\mathcal{Z}^{[XY]}_{\underline{a}(\underline{b})}(\beta) \geqslant \mathcal{Z}^{[XY]}_{\underline{a}(\underline{b}\vee\delta_u)}(\beta)\mathcal{Z}^{[XY]}_{\underline{a}(\underline{b}\vee\delta_v)}(\beta)$. Let us define η_u by $a_u(b_u)-\eta_u = \sqrt{1-(b_u+\epsilon_u)^2}$. Unless otherwise specified, the objects a_i are defined with respect to the reference configuration $\underline{b}$. Writing the XY spins in vector form: $(\hat{x}_i, \hat{y}_i) = \boldsymbol{t}_i$, the desired inequality amounts to showing that

$$\begin{aligned}
&\mathbb{E}_{\underline{a}}^{[XY]}\left(\exp\{\beta J_{u,v}\eta_u\eta_v \boldsymbol{t}_u \cdot \boldsymbol{t}_v\} \exp\left\{ -\beta \sum_i J_{u,i} a_i \eta_u \boldsymbol{t}_u \cdot \boldsymbol{t}_i \right\} \exp\left\{ -\beta \sum_j J_{v,j} a_j \eta_v \boldsymbol{t}_v \cdot \boldsymbol{t}_j \right\} \right) \\
&\qquad \geqslant \mathbb{E}_{\underline{a}}^{[XY]}\left(\exp\left\{ -\beta \sum_i J_{u,i} a_i \eta_u \boldsymbol{t}_u \cdot \boldsymbol{t}_i \right\} \right) \\
&\qquad \times \mathbb{E}_{\underline{a}}^{[XY]}\left(\exp\left\{ -\beta \sum_j J_{v,j} a_j \eta_v \boldsymbol{t}_v \cdot \boldsymbol{t}_j \right\} \right).
\end{aligned}$$

where $\mathbb{E}_{\underline{a}}^{[XY]}(-)$ denotes expectation with respect to the XY measure defined by the Hamiltonian with couplings $J_{i,j}a_i a_j$. Retaining only the terms that are lowest order in η_u, η_v, this reduces to showing that (i)

$$\mathbb{E}_{\underline{a}}^{[XY]}(\boldsymbol{t}_u \cdot \boldsymbol{t}_v) \geqslant 0$$

and that (ii)

$$\mathbb{E}_{\underline{a}}^{[XY]}((\boldsymbol{t}_u \cdot \boldsymbol{t}_i)(\boldsymbol{t}_v \cdot \boldsymbol{t}_j)) \geqslant \mathbb{E}_{\underline{a}}^{[XY]}(\boldsymbol{t}_u \cdot \boldsymbol{t}_i)\mathbb{E}_{\underline{a}}^{[XY]}(\boldsymbol{t}_v \cdot \boldsymbol{t}_j).$$

The first inequality is the standard Griffiths inequality (for rotors) and the second is the correlation inequality proved by Ginibre; both of these are proved in [G] (see also [MMP]). □

Corollary 1. Under the same conditions as lemma 2, the measures $\mathbb{M}_\beta(-)$ are FKG (have positive correlations).

Proof. Let $\mu_{\underline{b},2\beta}(-)$ denote the $q = 2$ random-cluster measures as described earlier. Explicitly

$$\mu_{\underline{b},2\beta}(\omega) \propto B_{\underline{b},2\beta}(\omega) 2^{c(\omega)}. \tag{4}$$

We may decompose the measure $\mathbb{M}_\beta$ according to

$$\mathbb{M}_\beta(-) = \int_{\underline{b}} \mathrm{d}\rho_\beta\,(\underline{b}) \mu_{\underline{b},2\beta}(-). \tag{5}$$

Let $\mathcal{A}$ and $\mathcal{B}$ denote increasing bond events. Then

$$\mathbb{M}_\beta(\mathcal{A} \cap \mathcal{B}) = \int_{\underline{b}} \mathrm{d}\rho_\beta\,(\underline{b}) \mu_{\underline{b},2\beta}(\mathcal{A} \cap \mathcal{B}) \geqslant \int_{\underline{b}} \mathrm{d}\rho_\beta\,(\underline{b}) \mu_{\underline{b},2\beta}(\mathcal{A}) \mu_{\underline{b},2\beta}(\mathcal{B}) \tag{6}$$

by the FKG property of the random-cluster measures. However, random-cluster probabilities of increasing events are increasing functions of all the couplings—and hence of the $\underline{b}$. Thus $\mu_{\underline{b},2\beta}(\mathcal{A})$ and $\mu_{\underline{b},2\beta}(\mathcal{B})$ (considered as functions of $\underline{b}$) are positively correlated and we conclude, by the result of lemma 2, that $\mathbb{M}_\beta(\mathcal{A} \cap \mathcal{B}) \geqslant \mathbb{M}_\beta(\mathcal{A})\mathbb{M}_\beta(\mathcal{B})$. □

Corollary 2. Consider adding to the isotropic zero-field Hamiltonian the following types of terms: (a) A magnetic field term: $\sum_i h_i \sigma_i b_i$ (i.e. $\boldsymbol{h}$ points in the $\hat{z}$-direction) with all the non-zero h_i of the same sign—say positive. (b) A term that modifies the coupling between the z-components, $\sum_{\langle i,j \rangle} K_{i,j} b_i b_j \sigma_i \sigma_j$ with $J_{i,j} + K_{i,j}$ non-negative. Then the conclusions of lemma 2 still hold. Furthermore, if one set of K and h is 'larger' (in the natural sense of partial order) than a second, the associated measures are correspondingly FKG ordered.

Proof. Neither of these terms have any effect on the XY portion in our proof of lemma 2; the remainder of what is needed is proved exactly as in [C]: proposition A.1 establishes the FKG property (when all the $J_{i,j} + K_{i,j}$ are non-negative) and similar considerations were shown to apply to non-zero (and non-negative) magnetic fields by considering 'ghost sites'. The stated FKG dominance was the corollary to this proposition. $\square$

Corollary 3. Let $\Lambda_1 \subset \mathbb{Z}^d$ denote a finite set. Then $\rho^w_{\beta;\Lambda_1}(-) \underset{\text{FKG}}{\geqslant} \rho^w_{\beta;\Lambda_1}(-)$ and similarly for the bond measures $\mathbb{M}_\beta(-)$. Furthermore let $\Lambda_2 \subset \Lambda_1$ and let $\rho^w_{\beta;\Lambda_1|\Lambda_2}(-)$ and $\mathbb{M}^w_{\beta;\Lambda_1|\Lambda_2}(-)$ denote the restrictions of the Λ_1 wired measures to Λ_2. then

$$\rho^w_{\beta;\Lambda_1|\Lambda_2}(-) \underset{\text{FKG}}{\leqslant} \rho^w_{\beta;\Lambda_2}(-)$$

and

$$\mathbb{M}^w_{\beta;\Lambda_1|\Lambda_2}(-) \underset{\text{FKG}}{\leqslant} \mathbb{M}^w_{\beta;\Lambda_2}(-).$$

Proof. As is not hard to see, the wired measures can be constructed from the free measure—or the measure on the larger space—by the addition to the Hamiltonian of some $K_{i,j}$ and/or h_i which are then taken to infinity. The stated FKG dominations follow from (the limiting version of) corollary 2. $\square$

An immediate consequence of corollary 3 is the existence of Π_∞ independent of the sequence (Λ_k)—this follows from standard monotonicity arguments. In addition, we have the following corollary.

Corollary 4. Consider a finite graph (with no boundary conditions) in particular some finite $\Lambda \subset \mathbb{Z}^d$ with free or periodic boundary conditions. Let $\langle - \rangle_\beta$ denote the corresponding Gibbs state for the zero-field Hamiltonian (equation (1)) or the infinite-volume limit thereof (defined, if necessary by subsequence) and $\mathbb{M}_\beta(-)$ the associated bond measure. Then

$$3\mathbb{M}_\beta(i \leftrightarrow j) \geqslant \langle \boldsymbol{s}_i \cdot \boldsymbol{s}_j \rangle_\beta \geqslant K^2 \mathbb{M}_\beta(i \leftrightarrow j)$$

where $i \leftrightarrow j$ is the event that the sites i and j are in the same connected cluster and K is a non-singular function of β and coordination number. On $\mathbb{Z}^d$, in the single-phase regime, similar bounds relate the susceptibility to the average cluster size.

Proof. This is exactly as proved in [C]. In brief: it is enough to examine $\langle s_i^z s_j^z \rangle_\beta = \langle b_i b_j \sigma_i \sigma_j \rangle_\beta$. Decomposing into clusters, it is not hard to see that there is vanishing contribution from any contribution in which i is not connected to j and that $\sigma_i = \sigma_j$ whenever it is. Hence the identity

$$\langle s_i^z s_j^z \rangle_\beta = V_\beta^W (b_i b_j \mathbb{I}_{\{i \leftrightarrow j\}}) \tag{7}$$

where $\mathbb{I}$ is an indicator. The upper bound is obtained by the observation $b_i b_j \leqslant 1$ and the lower bound by the FKG inequality and consideration of the worst case scenarios on the neighbours of i and j to estimate $\langle b_i \rangle_\beta$ and $\langle b_j \rangle_\beta$. The result on the susceptibility is obtained by summing the spin–spin correlation function. $\square$

In a similar fashion, the proof of theorem 1 follows the same lines as the corresponding result for the XY model.

Proof of theorem 1. The spontaneous magnetization is not smaller than the average of s_0^z in any limiting state. Hence the lower bound. To obtain the upper bound, we set $h_i \equiv h > 0$ and note that for a.e. h, $m(h)$ is independent of thermodynamic state. Hence we may employ wired boundary conditions at $h > 0$ and, with a little work, exchange the $h \downarrow 0$ and infinite-volume limits. Details can be found in [C, proof of theorem 4A]. $\square$

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